**Algebraic Geometry Lecture 17 – Categories and stuff**

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Motivating example: Sets and functions.

**Def$^{\mathbf{n}}$.** A category $\mathcal{C}$ is a collection of *objects*, $\mathrm{ob}(\mathcal{C})$, and *morphisms/maps/arrows*, $\mathrm{mor}(\mathcal{C})$, which each have a source and a target in $\mathrm{ob}(\mathcal{C})$. If $f \in \mathrm{mor}(\mathcal{C})$ and the source of $f$ is $s(f) = X$, and its target is $t(f) = Y$, then we write $f : X \to Y$.

- For each $c \in \mathrm{ob}(\mathcal{C})$, there is a unique distinguished morphism $\mathrm{id}_c$.
- We have composition: if $f : X \to Y$ and $g : Y \to Z$ then there is a composite $g \circ f : X \to Z$, subject to:
  - Associativity: for each $f, g, h \in \mathrm{mor}(\mathcal{C})$, we have $(f \circ g) \circ h = f \circ (g \circ h)$ whenever this is defined.
  - Identity: $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_Y \circ f = f$.

**Examples.**

| Category | Objects | Morphisms |
|---|---|---|
| Set | Sets | Functions |
| Grp | Groups | Group homomorphisms |
| Ab | Abelian groups | Group homomorphisms |
| Top | Topological spaces | Continuous maps |
| Toph | Topological spaces | Continuous maps up to homotopy |

The above examples all have sets as their objects. This is "nice" because their objects have elements, e.g. $x \in X \in \mathrm{ob}(\mathrm{Set})$. Such categories are called concrete.

**Other examples.**

A *poset* $(S, \leqslant)$ is a set $S$ with a partial ordering. We can describe this as a category $\mathcal{C}$. Let $\mathrm{ob}(\mathcal{C}) = S$ and non-identity morphisms be:

for $x, y \in S$ there is a unique morphism $f : x \to y$ if and only if $x \leqslant y$.

Composition follows from the transitive law for posets: if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

In sets we like knowing when a function is injective or surjective. We say a morphism $m : X \to Y$ is monic (like an injection) if for every $f : W \to X$ and $f' : W \to X$ we have $m \circ f = m \circ f' \Rightarrow f = f'$.

We call a morphism $e : X \to Y$ epi (surjective) if for every $f : Y \to Z$ and $f' : Y \to Z$ we have $f \circ e = f' \circ e \Rightarrow f = f'$.

If a morphism $f$ is both monic and epi then we call $f$ and *isomorphism.*

[1]Notes typed by Lee Butler based on a lecture given by Joe Grant. Any errors are the responsibility of the typist. Or the US sub-prime mortgage crisis.

**Remarks.**

Let $G$ be a group. Define a category $\mathcal{C}$ such that $\mathrm{ob}(\mathcal{C}) = \{*\}$ and $\mathrm{mor}(\mathcal{C}) = G$. So groups are one-object categories with invertible morphisms.

In the good old days ($\leqslant 1950$) we would write a function with domain $X$ and codomain $Y$ as $f(X) \subset Y$. The notation $f : X \to Y$ comes from category theory.

**Functors.**

**Def$^{\mathbf{n}}$.** A functor $F : \mathcal{C} \to \mathcal{D}$ for categories $\mathcal{C}$ and $\mathcal{D}$ is a function that takes $\mathrm{ob}(\mathcal{C})$ to $\mathrm{ob}(\mathcal{D})$ and $\mathrm{mor}(\mathcal{C})$ to $\mathrm{mor}(\mathcal{D})$, such that if $f : X \to Y$ then $F(f) : F(X) \to F(Y)$, and:

- $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all $c \in \mathrm{ob}(\mathcal{C})$.
- If $g, f \in \mathrm{mor}(\mathcal{C})$ then $F(g \circ f) = F(g) \circ F(f)$ in $\mathcal{D}$.

There are a lot of functors.

**E.g. 1.** The identity functor $F : \mathcal{C} \to \mathcal{C}$.

**E.g. 2.** $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$, the power set functor. Let $X \in \mathrm{ob}(\mathrm{Set})$, then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}.$$

Let $f : X \to Y$ be a function. To define $\mathcal{P}$ we need a function $F$ such that $F(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$. Let $X' \in \mathcal{P}(X)$, i.e. $X' \subseteq X$. Then define

$$F(f)(X') = f(X') \subseteq Y.$$

So $F(f)(X') \in \mathcal{P}(Y)$.

**E.g. 3.** Let $G, H$ be two groups and $\mathcal{C}_G, \mathcal{C}_H$ be the categories associated to them. Then a functor $F : \mathcal{C}_G \to \mathcal{C}_H$ is a homomorphism. There is only one object in each of $\mathrm{ob}(\mathcal{C}_G)$ and $\mathrm{ob}(\mathcal{C}_H)$ so there are no worries there. On morphisms,

$$F(f_g \circ f_{g'}) = F(f_g) \circ F(f_{g'})$$

is the same as

$$\varphi(gg') = \varphi(g)\varphi(g').$$

**Natural transformations.**

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors. We want to define a natural transformation "$F \Rightarrow G$" or "$F \dot{\to} G$".

We define the natural transformation, $\eta$. We want a map $F(c) \mapsto G(c)$ for each $c \in \mathrm{ob}(\mathcal{C})$. This map must be "nice". Define the map $F(c) \to G(c)$ by $\eta_c$. So a natural transformation is, for each $c \in \mathrm{ob}(\mathcal{C})$, an assignment $F(c) \to G(c)$ denoted by $\eta_c$ such that for all $f : c \to d$ in $\mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\eta_c} & G(c) \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
F(d) & \xrightarrow{\eta_d} & G(d)
\end{array}
$$